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Journal of APPLIED MATHEMATICS AND MECHANICS

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Journal of Applied Mathematics and Mechanics 69 (2005) 49–56

THE EFFECT OF TORSION ON THE STABILITY OF AN ELASTIC CYLINDER UNDER TENSION†

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(*Received* 12 *May* 2003)

The effect of torsion on instability in the form of the formation of a neck in a stretched rod in the shape of a circular cylinder is investigated. The stability is considered using the three-dimensional equations of neutral equilibrium of an isotropic incompressible solid. The subcritical state is described by the exact solution of the problem of the non-linear theory of elasticity regarding the equilibrium of a twisted and stretched cylinder. After separating the variables, the equations of neutral equilibrium are reduced to a system of ordinary differential equations, which are solved numerically. Numerical results are obtained for a Biderman material and a material with a power dependence of the specific energy on the deformation. A region of stability is constructed in the plane of the loading parameters, which are the longitudinal elongation and the angle of torsion. This region is compared with the region of convexity of the energy per unit length of the twisted and stretched cylinder.

1. THE EQUILIBRIUM OF A CYLINDER IN THE SUBCRITICAL STATE

The isochoric deformation of the torsion and extension (compression) of a continuous circular cylinder [1] made of isotropic incompressible elastic material, is given by the relations

$$R = \alpha^{-1/2}r, \quad \Phi = \varphi + \psi z, \quad Z = \alpha z, \quad \alpha, \psi = \text{const} \tag{1.1}$$

Here r, φ, z are cylindrical coordinates in the reference configuration, R, Φ, Z are cylindrical coordinates in the actual configuration, α is the extension coefficient along the cylinder axis, and ψ is the angle of torsion per unit length. The expressions for the measures of deformation and invariance, corresponding to (1.1), will have the following form

$$\begin{aligned} \mathbf{F} &= \alpha^{-1}\mathbf{e}_R\mathbf{e}_R + f(R)\mathbf{e}_\Phi\mathbf{e}_\Phi + \alpha\psi R(\mathbf{e}_\Phi\mathbf{e}_Z + \mathbf{e}_Z\mathbf{e}_\Phi) + \alpha^2\mathbf{e}_Z\mathbf{e}_Z \\ \mathbf{g} &= \alpha(\mathbf{e}_R\mathbf{e}_R + \mathbf{e}_\Phi\mathbf{e}_\Phi) - \psi R(\mathbf{e}_\Phi\mathbf{e}_Z + \mathbf{e}_Z\mathbf{e}_\Phi) + \alpha^{-1}f(R)\mathbf{e}_Z\mathbf{e}_Z \end{aligned} \tag{1.2}$$

$$I_1 = 2\alpha^{-1} + \alpha^2 + \psi^2R^2, \quad I_2 = 2\alpha + \alpha^{-2} + \alpha^{-1}\psi^2R^2, \quad f(R) = \alpha^{-1} + \psi^2R^2 \tag{1.3}$$

In these formulae $\mathbf{e}_R$, $\mathbf{e}_\Phi$, $\mathbf{e}_Z$ is the orthonormalized vector basis of cylindrical coordinates in the actual configuration, $\mathbf{g}$ is the Almansi strain measure, $\mathbf{F} = \mathbf{g}^{-1}$ is the Finger strain measure, and I_1 and I_2 are the first and second invariants of the tensor $\mathbf{F}$ (the third invariant is equal to unity in view of the incompressibility condition).

The constitutive relation for an isotropic incompressible elastic solid [1] has the form

$$\begin{aligned} \mathbf{T} &= \kappa_1(I_1, I_2)\mathbf{F} - \kappa_2(I_1, I_2)\mathbf{g} - p\mathbf{E} \\ \kappa_1(I_1, I_2) &= 2\frac{\partial W}{\partial I_1}, \quad \kappa_2(I_1, I_2) = 2\frac{\partial W}{\partial I_2} \end{aligned} \tag{1.4}$$

†*Prikl. Mat. Mekh.* Vol. 69, No. 1, pp. 53–60, 2005.

0021–8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2005.01.005

Here $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{E}$ is the unit tensor, p is the component of the stress in the incompressible solid, not determined by the strains, and $W(I_1I_2)$ is the specific potential energy of deformation of the elastic cylinder.

It follows from relations (1.2) and (1.4) and the equilibrium equations that the torsional stresses τ_{RZ} and $\tau_{R\Phi}$ are equal to zero, while the remaining components of the tensor $\mathbf{T}$ in the $\mathbf{e}_R$, $\mathbf{e}_\Phi$, $\mathbf{e}_Z$ basis depend only on the coordinate R.

In the case when the side surface of the cylinder is unloaded, the expressions for the stresses and the function p have the form

$$\begin{aligned}
&\sigma_R = -\psi^2\int_R^{R_0}\kappa_1(\rho)\rho d\rho, \quad R_0 = \alpha^{-1/2}r_0, \quad \sigma_\Phi = \sigma_R + \psi^2\kappa_1(R)R^2 \\
&\sigma_Z = (\alpha^2-\alpha^{-1})\kappa_1(R) + [\alpha - \alpha^{-1}f(R)]\kappa_2(R) + \sigma_R \\
&\tau_{\Phi Z} = \psi R[\alpha\kappa_1(R) + \kappa_2(R)], \quad p(R) = \alpha^{-1}\kappa_1(R) - \alpha\kappa_2(R) - \sigma_R
\end{aligned} \tag{1.5}$$

where r_0 and R_0 are the radii of the cylinder before and after deformation respectively. The values of the functions $\kappa_i(R)$ are obtained after substituting relations (1.3) into the expressions for $\kappa_i(I_1, I_2)$.

The stresses acting in any transverse section of the cylinder reduce to a longitudinal force P and a torsional moment M. For a specified material, the quantities P and M will be certain known functions of the parameters α and ψ, defined by the formulae

$$P(\alpha,\psi) = 2\pi\int_0^{R_0}\sigma_Z(R)RdR, \quad M(\alpha,\psi) = 2\pi\int_0^{R_0}\tau_{\Phi Z}(R)R^2dR \tag{1.6}$$

2. THE LINEARIZED EQUILIBRIUM EQUATIONS

We will consider a perturbed equilibrium state, infinitesimally close to subcritical and existing under the same external conditions. The position of the particles of the solid in the perturbed state is given by the radius vector $\mathbf{R} + \eta\mathbf{w}$, where $\mathbf{R}$ is the radius vector in the subcritical state, $\mathbf{w}$ is the vector of additional displacements, and η is a small parameter. The linearized equilibrium equations of the incompressible solid, which describes the perturbed state and also the so-called equations of neutral equilibrium, have the form [1, 2]

$$\nabla\cdot\mathbf{\Theta} = 0, \quad \mathbf{\Theta} = \mathbf{T}^{\bullet} - (\nabla\mathbf{w})^T\cdot\mathbf{T}, \quad \mathbf{T}^{\bullet} = \left[\frac{d}{d\eta}\mathbf{T}(\mathbf{R}+\eta\mathbf{w})\right]_{\eta=0} \tag{2.1}$$

$$\nabla\cdot\mathbf{w} = 0; \quad \nabla = \mathbf{e}_R\frac{\partial}{\partial R} + \mathbf{e}_\Phi\frac{\partial}{R\partial\Phi} + \mathbf{e}_Z\frac{\partial}{\partial Z} \tag{2.2}$$

Here $\mathbf{\Theta}$ is the linearized Piola stress tensor and ∇ is the nabla operator in the subcritical deformation configuration of the cylinder. Equation (2.2) is the linearized condition of incompressibility. The linearized boundary conditions on the side surface of the cylinder

$$\mathbf{e}_R\cdot\mathbf{\Theta} = 0 \quad \text{for} \quad R = R_0 \tag{2.3}$$

express the fact that there is no surface load in the perturbed state.

Taking relation (1.4) into account, the expression for the tensor $\mathbf{\Theta}$ becomes [3]

$$\begin{aligned}
\mathbf{\Theta} = {} & \kappa_1(R)\mathbf{F}\cdot\nabla\mathbf{w} + \kappa_2(R)[(\nabla\mathbf{w})\cdot\mathbf{g} + \mathbf{g}\cdot(\nabla\mathbf{w})^T + (\nabla\mathbf{w})^T\cdot\mathbf{g}] + \\
& + p(R)(\nabla\mathbf{w})^T + \kappa_1^{\bullet}\mathbf{F} - \kappa_2^{\bullet}\mathbf{g} + q\mathbf{E}
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
&\kappa_i^{\bullet} = 2\kappa_{i1}(R)\mathrm{tr}(\mathbf{F}\cdot\nabla\mathbf{w}) - 2\kappa_{i2}(R)\mathrm{tr}(\mathbf{g}\cdot\nabla\mathbf{w}) \\
&\kappa_{ij} = \frac{d\kappa_i}{dI_j}, \quad q = -p^{\bullet}; \quad i, j = 1, 2
\end{aligned} \tag{2.5}$$

To determine the function $\kappa_{ij}(R)$ we must substitute the values of the invariants of the Finger strain measure **F** into the expressions for the derivatives $\partial\kappa_i(I_1, I_2)/\partial I_j$.

We will represent the vector **w** in the basis of cylindrical coordinates as follows:

$$\mathbf{w} = u\mathbf{e}_R + \upsilon\mathbf{e}_R + w\mathbf{e}_Z$$

Substituting expression (2.4) into Eq. (2.1) and adding condition (2.2), we obtain a system of four partial differential equations in the four unknown functions u, υ, w and q. This solution allows of solutions of the form

$$\begin{aligned} u &= U(R)\cos(n\Phi+\lambda Z), \quad \upsilon = V(R)\sin(n\Phi+\lambda Z) \\ w &= W(R)\sin(n\Phi+\lambda Z), \quad q = Q(R)\cos(n\Phi+\lambda Z); \quad n = 0, 1, 2, \ldots \end{aligned} \tag{2.6}$$

Here λ is a real number. The integral parameter n defines the form of the loss of stability of the cylinder. For example, the case $n = 0$ corresponds to an axisymmetric form of the loss of stability, while $n = 1$ corresponds to the rod form. Other values of this parameter correspond to forms of more complex structure.

Using relations (2.6) in the equations of neutral equilibrium and the boundary conditions (2.3), the variables Φ and Z can be separated, and we obtain a boundary-value problem for the system of four ordinary differential equations in U, V, W and Q.

If we put $\lambda = \pi m/l$ $(m = 1, 2, \ldots)$, where l is the length of the deformed cylinder, then, as was shown earlier in [3], the solutions (2.6) will correspond to the following conditions of the clamping of the ends of the cylinder:

when $n = 0$ the cylinder is fastened by two smooth plates;

when $n = 1$ there is a hinged support for displacements in the direction of the y axis and a sliding clamp for displacements along the x axis.

3. A BIDERMAN MATERIAL

We will consider the model of a Biderman material [4], the specific potential energy of deformation of which is given by the relation

$$\begin{aligned} W(I_1, I_2) &= d_0(I_2-3) + d_1(I_1-3) + d_2(I_1-3)^2 + d_3(I_1-3)^3 \\ d_0, d_1, d_2, d_3 &= \text{const} \end{aligned} \tag{3.1}$$

In this case

$$\kappa_1(I_1, I_2) = d_1 + 2d_2(I_1-3) + 3d_3(I_1-3)^2, \quad \kappa_2(I_1, I_2) = d_0$$

The sufficient conditions for the Hadamard condition to be satisfied for this material have the form [5, 6]

$$d_0 \geq 0, \quad d_1 \geq 0, \quad d_3 \geq 0, \quad d_1 + d_3 \geq 0, \quad 3d_2 + \sqrt{15 d_1 d_3} \geq 0 \tag{3.2}$$

Consider the case when $d_0 = 0$. Then $\kappa_2 \equiv 0$, and the system of equations (2.1), (2.2), taking relations (2.6) into account, can be written as follows:

$$\left(p + \frac{\kappa_1}{\alpha} + \frac{2\kappa_{11}}{\alpha^2}\right)U'' + \left(p' + \frac{p}{R} + \frac{\kappa_1}{\alpha R} + \frac{\kappa_1'}{\alpha} + \frac{2\kappa_{11}}{\alpha^2 R} + \frac{2\kappa_{11}'}{\alpha^2}\right)U' +$$

$$+\left(-\frac{p}{R^2} - \left[\frac{f}{R^2} + \frac{nf_1}{R} + \alpha\lambda\gamma\right]\kappa_1 + 2\left[\frac{f'}{\alpha} - \frac{f^2}{R}\right]\frac{\kappa_{11}}{R} + \frac{2f\kappa_{11}'}{\alpha R}\right)U +$$

$$+\left(\frac{np}{R} + \frac{2f_1\kappa_{11}}{\alpha}\right)V' + \left(-\frac{np}{R^2} - \frac{2f_1\kappa_1}{R} + 2\left[\frac{f_1'}{\alpha} + \frac{f_1}{\alpha R} - \frac{f_1 f}{R}\right]\kappa_{11} + \frac{2f_1\kappa_{11}'}{\alpha}\right)V +$$

$$
\begin{gathered}
+(p\lambda+2\gamma\kappa_{11})W'+2\gamma\left([1-\alpha f]\frac{\kappa_{11}}{R}+\kappa_{11}'\right)W+Q'=0 \\
-\left(\frac{np}{R}+\frac{2f_1\kappa_{11}}{\alpha}\right)U'-\left(\frac{np'}{R}+\frac{np}{R^2}+\frac{2f_1\kappa_1}{R}+\frac{2f_1f\kappa_{11}}{R}\right)U+\frac{\kappa_1}{\alpha}V''+ \\
+\left(\frac{\kappa_1}{\alpha R}+\frac{\kappa_1'}{\alpha}\right)V'-\left(\frac{p'}{R}+\frac{n^2p}{R^2}+\left[\frac{f}{R^2}+\frac{nf_1}{R}+\alpha\lambda\gamma\right]\kappa_1+2f_1^2\kappa_{11}\right)V-\left(\frac{np\lambda}{R}+2\alpha\gamma f_1\kappa_{11}\right)W-\frac{n}{R}Q=0 \\
-(p\lambda+2\gamma\kappa_{11})U'-\left(p'\lambda+\frac{p\lambda}{R}+\frac{2\alpha\gamma f\kappa_{11}}{R}\right)U-\left(\frac{np\lambda}{R}+2\alpha\gamma f_1\kappa_{11}\right)V+ \\
+\frac{\kappa_1}{\alpha}W''+\left(\frac{\kappa_1}{\alpha R}+\frac{\kappa_1'}{\alpha}\right)W'-\left(p\lambda^2+\left[\frac{nf_1}{R}+\alpha\lambda\gamma\right]\kappa_1+2\alpha^2\gamma^2\kappa_{11}\right)W-\lambda Q=0 \\
U'+\frac{1}{R}U+\frac{n}{R}V+\lambda W=0 \\
\gamma=\psi n+\alpha\lambda,\quad f_1(R)=\frac{f(R)n}{R}+\alpha\psi R\lambda
\end{gathered}
\tag{3.3}
$$

The boundary conditions on the side surface $R = R_0$ take the form

$$
\begin{gathered}
\left(p+\frac{\kappa_1}{\alpha}+\frac{2\kappa_{11}}{\alpha^2}\right)U'+\frac{2f\kappa_{11}}{\alpha R_0}U+\frac{2f_1\kappa_{11}}{\alpha}V+2\gamma\kappa_{11}W+Q=0 \\
\frac{pn}{R_0}U-\frac{\kappa_1}{\alpha}V'+\frac{p}{R_0}V=0,\quad p\lambda U-\frac{\kappa_1}{\alpha}W'=0
\end{gathered}
\tag{3.4}
$$

Eliminating the unknown functions W and Q from Eqs (3.3), we obtain a system of six first-order equations, written in matrix form

$$
\mathbf{X}'+\mathbf{A}(R)\mathbf{X}=0,\quad \mathbf{X}=\begin{cases}\{U, U', U'', U''', V, V'\}, & n=0\\ \{U, U', U'', V, V', V''\}, & n>0\end{cases}
\tag{3.5}
$$

where $\mathbf{X}$ is a column vector of the unknown functions and $A(R)$ is a matrix of the coefficients of the system.

To solve system (3.5) we need three more boundary conditions, which can be obtained by requiring that the functions U, V, W and Q and their derivatives are bounded for $R = 0$ (in the case when $n > 2$ we can take any three conditions of the four, indicated in (3.6)):

$$
\begin{gathered}
n=0:\ U(0)=U''(0)=V(0)=0 \\
n=1:\ U(0)+V(0)=0,\quad U'(0)=V'(0)=0 \\
n=2:\ U(0)=V(0)=0,\quad U'(0)+V'(0)=0 \\
n>2:\ U(0)=V(0)=U'(0)=V'(0)=0
\end{gathered}
\tag{3.6}
$$

System (3.5) can be solved by the finite-difference method [3].

The results presented below were obtained for a cylinder with a diameter/length ratio of 0.1 in the undeformed state. All the curves are symmetrical about the $\psi_0 = 0$ axis, and hence only the region $\psi_0 \geq 0$ is shown. On each curve we show the value of the parameter m to which it corresponds.

In Fig. 1 we show bifurcation curves in the plane of the parameters α and $\psi_0 = \psi r_0$ for a Biderman material with the set of constants $d_0 = 0$, $d_1 = 27$, $d_2 = -60$ and $d_3 = 80$ for an axisymmetric form of loss of stability ($n = 0$) and for a rod form of instability ($n = 1$). No loss of stability of more complex form ($n > 1$) is observed for this material. In the axisymmetric case bifurcation of the equilibrium only occurs for the first few values of the parameter m (in this case for $m = 1, 2, \ldots, 16$), whereas for the

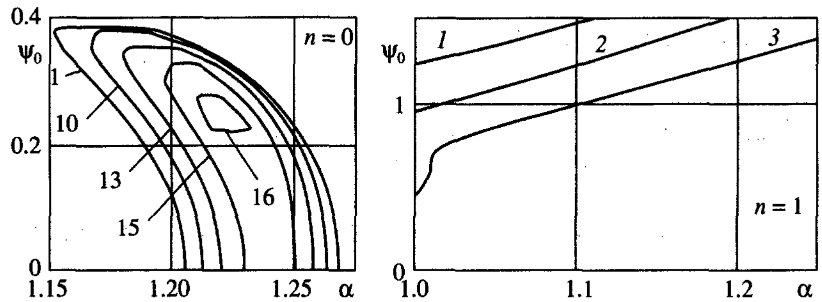

Fig. 1

rod form of loss of stability the bifurcation curves exist for any m. Intersections of the curves with the α axis give the bifurcation points for simple extension of the cylinder.

For simple extension, we obtain only the axisymmetric form of loss of stability (the formation of a neck), where all the bifurcation points lie on the descending part of the diagram of the stretching force P against the extension coefficient α.

This fact is not fortuitous, but is a property characteristic not only of the Biderman model but also of any incompressible isotropic material. The following theorem holds.

Theorem. Suppose the following conditions are satisfied:

(1) an isotropic incompressible material satisfies the Hadamard condition [1];

(2) a set Γ of the semi-axis $\alpha > 1$ exists, on which the stretching force $P(\alpha)$ is a non-decreasing function of the axial lengthening of the cylinder.

Then, where $\alpha \in \Gamma$ the second variation of the potential energy of the elastic cylinder is strictly positive on any virtual non-rigid shift from a state of simple extension

$$R = \alpha^{-1/2}r, \quad \Phi = \varphi, \quad Z = \alpha z$$

which denotes that there are non-trivial solutions of the linearized boundary-value problem (2.1)–(2.4) when $\psi = 0$ and stability of the stretched cylinder to small perturbations.

Proof. Expression (1.6) for the stretching force, taking solution (1.5) into account, can be written as

$$P = \pi r_0^2(\alpha - \alpha^{-2})(\kappa_1 + \alpha^{-1}\kappa_2) \tag{3.7}$$

Hence, by finding the derivative $dP/d\alpha$, we obtain the condition for the stretching force $P(\alpha)$ to be non-decreasing

$$\left(\alpha^2 + \frac{2}{\alpha}\right)\kappa_1 + \frac{3}{\alpha^2}\kappa_2 + 2\left(\alpha - \frac{1}{\alpha^2}\right)^2(\alpha^2\kappa_{11} + 2\alpha\kappa_{12} + \kappa_{22}) \geq 0 \tag{3.8}$$

The second variation of the potential energy of the elastic body has the form [2]

$$\delta^2\Pi = \frac{d^2}{d\eta^2}\iiint_V W(\mathbf{R} + \eta\mathbf{w})dV|_{\eta=0} = \iiint_V \mathbf{\Theta}\cdot\cdot\nabla\mathbf{w}^T dV$$

Hence, using relations (1.2), (1.3), (2.4) and (2.5), we obtain

$$\delta^2\Pi = \iiint_V \left[\left(\frac{\kappa_1}{\alpha} + \alpha\kappa_2\right)\left(\left(\frac{\partial w}{\partial Z} + 2\frac{\partial u}{\partial R}\right)^2 + \left(\frac{\partial v}{\partial R} - \frac{v}{R} + \frac{\partial u}{R\partial\Phi}\right)^2\right) + \right.$$
$$+ \frac{1}{\alpha}\left(\kappa_1 + \frac{\kappa_2}{\alpha}\right)\left(\left(\frac{\partial w}{\partial R} + \frac{\partial u}{\partial Z}\right)^2 + \left(\frac{\partial w}{R\partial\Phi} + \frac{\partial v}{\partial Z}\right)^2 + (\alpha^3 - 1)\left[\left(\frac{\partial u}{\partial Z}\right)^2 + \left(\frac{\partial v}{\partial Z}\right)^2\right]\right) +$$
$$\left. + \left(\left(\alpha^2 + \frac{2}{\alpha}\right)\kappa_1 + \frac{3}{\alpha^2}\kappa_2 + 2\left(\alpha - \frac{1}{\alpha^2}\right)^2(\alpha^2\kappa_{11} + 2\alpha\kappa_{12} + \kappa_{22})\right)\left(\frac{\partial w}{\partial Z}\right)^2\right] dV \tag{3.9}$$

According to results obtained previously [5], the inequalities

$$\kappa_1 + \alpha^{-1}\kappa_2 \geq 0, \quad \alpha^{-1}\kappa_1 + \alpha\kappa_2 \geq 0$$

follow from the Hadamard condition, where $\kappa_1 + \alpha^{-1}\kappa_2 \geq 0$ can only be equal to zero when $\alpha = 1$, which can be seen from expression (3.7) for the stretching force. Then, from expression (3.9) when $\alpha \in \Gamma$ (i.e. when condition (3.8) is satisfied) we obtain that $\delta^2\Pi \geq 0$ for any w, where the equality is only possible for vector fields with components of the form

$$u = u_0 + u_1 R + u_2 \Phi, \quad \upsilon = \upsilon_0 + \upsilon_1 R + \upsilon_2 \Phi, \quad w = w_0 + w_1 Z$$

where u_i, υ_i, w_j ($i = 0, 1, 2; j = 0, 1$) are certain constants. But, since the displacement field $\mathbf{w}$ must satisfy incompressibility condition (2.2) and the kinematic constraint $\mathbf{e}_R \cdot \mathbf{w}/_{R=R_0} = 0$, while the components u and υ must satisfy the condition $u|_{R=0} = \upsilon|_{R=0} = 0$, we must have $u_0 = u_1 = u_2 = \upsilon_0 = \upsilon_2 = w_1 = 0$, i.e. $\mathbf{w} = \{0, \upsilon_1 R, w_0\}$, which corresponds to rigid displacement and rotation of the body, which is of no interest here.

The theorem is proved.

In the case of the plane deformation of a rectangular beam made of incompressible isotropic material, it was proved previously [6] that there are no mixed forms of equilibrium on the rising part of the rod stretching diagram.

For a cylinder of Biderman material with the above-mentioned values of the constants d_0, d_1, d_2 and d_3, the set Γ consists of the section $1 \leq \alpha \leq 1.206$ and the ray $\alpha \geq 1.206$. The section $1.206 \leq \alpha \leq 1.269$ corresponds to the falling part of the stretching diagram, on which, according to Fig. 1, a loss of stability is possible.

The potential energy per unit length of the stretched and twisted rod is given by the formula

$$\Pi_0(\alpha, \psi) = 2\pi \int_0^{r_0} rW(I_1, I_2)dr \tag{3.10}$$

Expressions (1.3), which follow from (1.1), must here be substituted as I_1 and I_2. Using (1.4), (1.6) and (3.10), and also solution (1.5) of the problem of the torsion and stretching of a circular cylinder, we can prove that the following relations hold

$$P(\alpha, \psi) = \partial\Pi_0(\alpha, \psi)/\partial\alpha, \quad M(\alpha, \psi) = \partial\Pi_0(\alpha, \psi)/\partial\psi \tag{3.11}$$

Using expressions (3.11), the condition for strict convexity of the energy per unit length of the rod as a function of the axial extension and the angle of torsion can be represented in the form of the Drucker postulate [7], i.e. the requirement that the work of the increments of the generalized forces on small increments of generalized displacements must be positive

$$dPd\alpha + dMd\psi > 0 \tag{3.12}$$

For simple extension, when $\psi = 0$, inequality (3.12) is satisfied on the rising parts of the stretching diagram and is equivalent to the condition that the function of one variable, $\Pi_0(\alpha)$, is strictly convex. As was established above, loss of stability of the stretched cylinder is only possible on the falling parts of the stretching diagram, i.e. outside the region where the energy per unit length is convex. In this connection, it is of interest in the more general case of a twisted and stretched cylinder, to compare the region of strict convexity of the function $\Pi_0(\alpha, \psi)$ in the plane of the parameters α, ψ with the region of stability determined by solving the linearized homogeneous boundary-value problem (2.1)–(2.4). This comparison is shown in Fig. 2, where the dark region is the region of instability, while the region in which the potential energy ceases to be convex is shown hatched. It should be noted that the hatched region is somewhat wider than the closed region of axisymmetric instability and completely encloses it. The region in which the convexity of the energy breaks down and the region of axisymmetric instability differ very little and hence are indistinguishable in Fig. 2.

For combined loading of the cylinder by a stretching force and a torsional moment, the axisymmetric forms ($n = 0$) of equilibrium bifurcation can only exist in the hatched region, where the property of energy convexity $\Pi_0(\alpha, \psi)$ is lost. However, the rod form of instability ($n = 1$), as can be seen from Fig. 2, occurs in the region where the function $\Pi_0(\alpha, \psi)$ is convex.

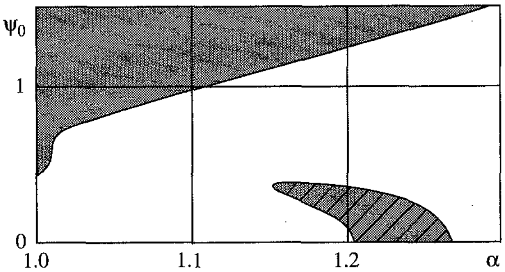

Fig. 2

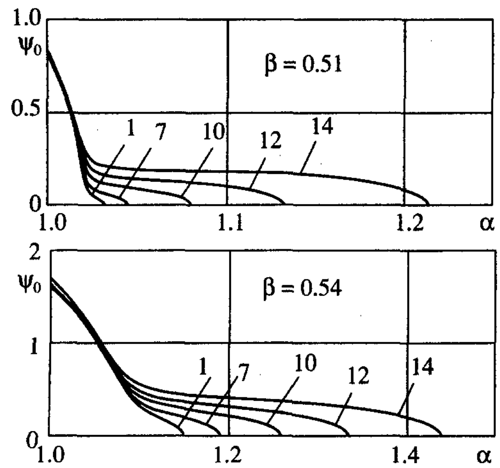

Fig. 3

Analysing the stability region, we can conclude that, using torsion, one can circumvent the instability region and achieve greater elongation without loss of stability than for simple extension. For cylinders with a different diameter/length ratio the stability region differs only slightly from that presented above.

4. A POWER-LAW MATERIAL

We will consider the model of an incompressible material proposed previously in [8]. The specific potential energy of deformation of this material is given by the relation

$$W(I_1, I_2) = d(I_1 - 3)^\beta, \quad d>0, \quad \beta \geq 1/2 \tag{4.1}$$

For values of β close to $1/2$, the stretching diagram of the material (4.1) differs very little from the diagram of a rigidly plastic solid.

The value of the material constant d does not affect the stability region in the plane of the parameters α, ψ_0, so we will henceforth assume $d = 1$.

As calculations show, for the given model of the material only the axisymmetric form of loss of stability exist. In Fig. 3 we show bifurcation curves for the case when $\beta = 0.51$ and $\beta = 0.45$. Unlike a Biderman material, for this model in the axisymmetric case, equilibrium bifurcation occurs for any value of the parameter m.

Comparing the region of convexity of the potential energy per unit length and the stability region presented in Fig 4, for $\beta = 0.54$, we see that for large angles of torsion in the part of the region where the energy is convex, one obtains the axisymmetric form of loss of stability. At the same time, these regions almost coincide, and hence the region of energy convexity, which is fairly easy to construct, can be used as an approximation of the stability region. It is worth nothing that the accuracy of the approximation for large angles of torsion decreases as the power β increases.

Analysing the stability region we can conclude that for this model of the material, unlike a Biderman material, the greatest elongation of the cylinder without loss of stability occurs for pure extension, while additional torsion only gives rise to a loss of stability at an earlier stage of extension.

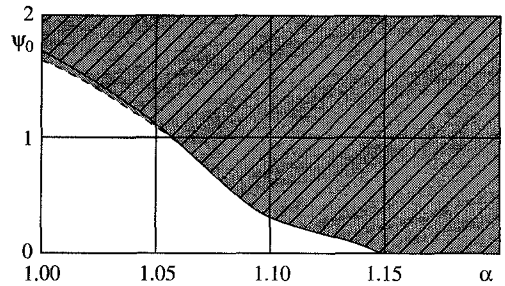

Fig. 4

For cylinders with a different diameter/length ratio the stability region, as in the case of a Biderman material, differs only slightly from these presented above.

This research was supported by the Russian Foundation for Basic Research (05-01-00638) and the "State Support for the Leading Scientific Schools" programme (NSh-2113.2003.1).

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Translated by R.C.G.